

## Landauer's principle in the quantum regime

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We demonstrate the validity of Landauer's erasure principle in the strong coupling quantum regime by treating the system-reservoir interaction in a thermodynamic way. We show that the initial coupling to the reservoir modifies both the energy and the entropy of the system, and provide explicit expressions for the latter for a damped quantum harmonic oscillator. These contributions are related to the Hamiltonian of mean force and dominate in the strong damping limit. They need therefore to be fully taken into account in any low temperature thermodynamic analysis of quantum systems.

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Information erasure is necessarily a dissipative process. According to Landauer's principle [1], erasure of one bit of information requires a minimum dissipation of heat of  $kT \ln 2$ , where  $T$  is the temperature and  $k$  the Boltzmann constant. As a result, the entropy of the environment increases by at least  $k \ln 2$ . The erasure principle establishes a fundamental relationship between information theory and thermodynamics. As such, it has played a pivotal role in the resolution of Maxwell's demon paradox [2]. The Landauer principle has been shown to hold for classical systems in the limit of strong [1] and weak [3,4] interaction with the external reservoir, as well as for weakly damped quantum systems [4]. However, its validity has recently been challenged in the strongly coupled quantum regime [5–8]. It has been claimed that in the latter the Clausius inequality for the von Neumann entropy,  $Q \leq kT \Delta S$ , may break down due to entanglement of system and reservoir at very low temperatures. This implies that information may be erased, that is, entropy ( $S$ ) decreased, while heat ( $Q$ ) is absorbed [9,10], in clear opposition to the Landauer principle. This supposed violation has been reported in a growing number of recent books [11–13] and reviews [14]. Meanwhile, several key results of quantum information theory have been derived with the help of the erasure principle. Important examples include the Holevo bound on accessible information [15], the no-cloning theorem [16], and the upper bound on the efficiency of entanglement distillation [17]. A failure of Landauer's erasure principle deep in the quantum domain would therefore have far-reaching consequences.

In this Rapid Communication, we resolve this quantum conundrum and show that the Landauer principle *does* hold in the strongly coupled quantum regime. We therefore establish the validity of the erasure principle in classical and quantum physics, for arbitrary reservoir interaction strengths.

The Clausius inequality asserts that, for a system initially in a Gibbs state, the change of entropy is always larger than or equal to the amount of heat received by the system divided by the temperature [18]; it is regarded as a general formulation of the second law of thermodynamics. Starting from a microscopic model for a strongly damped harmonic oscillator [see Eq. (1) below], it has been shown that the Clausius inequality is apparently violated at low temperatures during a quasistatic variation of the mass of the oscillator [5–10]. In order to understand this discrepancy, it is useful

to keep in mind that standard thermodynamics assumes that the system-reservoir coupling is negligibly small [18]. In this limit, the damped quantum system asymptotically relaxes to the correct thermal Gibbs state [19]. By contrast, for any finite interaction strength, the stationary state of the system is not of the Gibbs form because of the noncommutation of position and momentum [20]. Deviations from thermodynamics are therefore to be expected. It has, for instance, been observed that, at zero temperature, the oscillator is in an excited, mixed state and not in its pure ground state [20–22]. In the following, we resolve this paradoxical situation by considering, unlike in Refs. [5–10], the combined effect of the mass variation *and* of the coupling to the reservoir. By treating the initial coupling as a proper thermodynamic transformation, we explicitly show that it modifies both the energy and the entropy (i.e., information content) of the oscillator. In the limit of strong coupling, these contributions dominate those stemming from the variation of the mass. By considering the combined state transformation, we are able to demonstrate the general validity of the Clausius inequality for the von Neumann entropy, and hence of the Landauer principle.

*Microscopic system-reservoir model.* Following Refs. [5–10], we base our study of the Clausius inequality in the quantum domain on the standard model for quantum dissipation [23]. The latter consists of a harmonic oscillator linearly coupled to a bath of harmonic oscillators,

$$H = H_S + \sum_{j=1}^N \left[ \frac{p_j^2}{2m_j} + \frac{m_j \omega_j^2}{2} \left( x_j - \frac{C_j q}{m_j \omega_j^2} \right)^2 \right], \quad (1)$$

where the  $C_j$ 's are coupling constants. The Hamiltonian of the system is  $H_S = p^2/(2M) + M\omega^2 q^2/2$  with the usual notation. The frequencies of the reservoir modes and their coupling to the system are characterized by the Ohmic spectral density function,  $J(\nu) = (\pi/2) \sum_j C_j^2 / (m_j \omega_j) \delta(\nu - \omega_j) = \eta \nu \omega_D^2 / (\nu^2 + \omega_D^2)$ , with damping coefficient  $\eta$  and Debye cutoff frequency  $\omega_D$  [23]. Examples of damped oscillators are nanomechanical resonators that have been very recently cooled to the ground state [24]. They have been shown to be mass sensors of unprecedented sensitivity ( $10^{-24}$  kg and smaller) [25], enabling them to detect the tiny mass changes induced by single protein molecules attaching to them [26].

Let us assume that the system and the reservoir are initially decoupled and each is in equilibrium at temperature  $T$ . When the coupling is quasistatically increased, the combined system equilibrates to the Gibbs state  $\rho = \exp(-\beta H)/\text{Tr} \exp(-\beta H)$ . The reduced stationary phase space distribution of the oscillator is, however, non-Gibbsian: For the Ohmic model (1), it is given by a Gaussian with position and momentum dispersions [27],

$$\langle q^2 \rangle(\eta, M) = \frac{\hbar}{M\pi} \sum_{i=1}^3 \left[ \frac{(\lambda_i - \omega_D) \psi\left(1 + \frac{\beta\hbar\lambda_i}{2\pi}\right)}{(\lambda_{i+1} - \lambda_i)(\lambda_{i-1} - \lambda_i)} \right] + \frac{1}{M\beta\omega^2}, \quad (2)$$

$$\langle p^2 \rangle(\eta, M) = \frac{\hbar\eta\omega_D}{\pi} \sum_{i=1}^3 \left[ \frac{\lambda_i \psi\left(1 + \frac{\beta\hbar\lambda_i}{2\pi}\right)}{(\lambda_{i+1} - \lambda_i)(\lambda_{i-1} - \lambda_i)} \right] + M^2\omega^2 \langle q^2 \rangle(\eta, M). \quad (3)$$

The parameters  $\lambda_i(\eta, M)$  are the characteristic frequencies of the damped oscillator and  $\psi$  denotes the digamma function. In the limit of large cutoff frequency,  $\omega_D \gg \eta/M, \omega$ ,  $\lambda_{1,2}(\eta, M) \simeq \eta/(2M) \pm \zeta$ , with  $\zeta = [\eta^2/(4M^2) - \omega^2]^{1/2}$  and  $\lambda_3(\eta, M) = \omega_D - \eta/M$  [27]. To facilitate the discussion, we have here explicitly indicated the dependence on the coupling constant  $\eta$  and on the mass  $M$ . For finite coupling ( $\eta > 0$ ), system and reservoir are correlated, the variances are squeezed,  $M\omega^2 \langle q^2 \rangle(\eta, M) < \langle p^2 \rangle(\eta, M)/M$ , and the reduced stationary state is thus non-Gibbsian. For an uncoupled oscillator ( $\eta = 0$ ), Eqs. (2) and (3) reduce to their thermodynamic expressions,  $\langle q^2 \rangle(0, M) = \hbar/(2M\omega) \coth(\beta\hbar\omega/2)$ ,  $\langle p^2 \rangle(0, M) = \hbar M\omega/2 \coth(\beta\hbar\omega/2)$ , equipartition therefore holds, and the reduced state is of the Gibbs form.

*The paradox.* Suppose the coupled oscillator undergoes a quasistatic mass variation from  $M_0$  to  $M_1$ , as discussed in Refs. [5–10]; think of single molecules getting attached to a nanomechanical oscillator. We define the internal energy of the system as the stationary expectation value of its energy,  $U = \langle H_S \rangle = \text{Tr}_S[\rho_S H_S] = \langle p^2 \rangle/(2M) + M\omega^2 \langle q^2 \rangle/2$ , where  $\rho_S = \text{Tr}_B \rho$  is the reduced density operator of the oscillator. For an infinitesimal mass change  $dM$ , the oscillator exchanges an amount of heat with the external reservoir given by  $\delta Q^{(M)} = \text{Tr}_S[d\rho_S H_S]$ . As a result [5–10],

$$Q^{(M)} = \int_{M_0}^{M_1} \left( \frac{1}{2M} \frac{\partial \langle p^2 \rangle}{\partial M} + \frac{M\omega^2}{2} \frac{\partial \langle q^2 \rangle}{\partial M} \right) dM. \quad (4)$$

At the same time, the von Neumann entropy of the quantum oscillator changes by  $\Delta S^{(M)} = S(\eta, M_1) - S(\eta, M_0)$ , where  $S = -\text{Tr} \rho_S \ln \rho_S$  can be expressed as [5–10]

$$S = \left(v + \frac{1}{2}\right) \ln \left(v + \frac{1}{2}\right) - \left(v - \frac{1}{2}\right) \ln \left(v - \frac{1}{2}\right). \quad (5)$$

Here  $v = v(\eta, M) = \sqrt{\langle q^2 \rangle \langle p^2 \rangle} / \hbar$  denotes the phase space volume. The temperature dependence of  $\Delta^{(M)} = Q^{(M)} - kT \Delta S^{(M)}$  is shown in Fig. 1. We observe that  $\Delta^{(M)}$  is positive at very low temperatures, in apparent violation of the Clausius inequality which would require that  $\Delta^{(M)} \leq 0$  when  $\Delta M = M_1 - M_0 \geq 0$ .

Analytic expressions for  $Q^{(M)}$  and  $\Delta S^{(M)}$  (and hence  $\Delta^{(M)}$ ) can be derived close to zero temperature. A lowest order

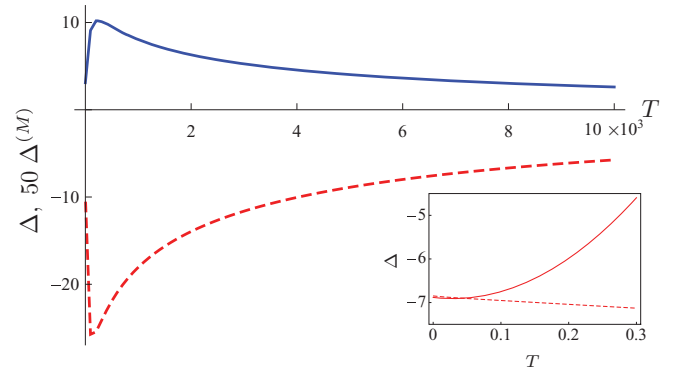


FIG. 1. (Color online) Temperature dependence of the quantities  $\Delta^{(M)} = Q^{(M)} - kT \Delta S^{(M)}$  (blue solid line) for the mass variation alone and  $\Delta = Q - kT \Delta S$  (red dashed line) for the combined transformation mass variation plus initial coupling. The former is positive, indicating an apparent violation of the Clausius inequality, while there is no violation for the combined transformation. The inset shows a comparison between the exact  $\Delta$  (red dashed line) and its low temperature approximation [Eq. (13)] (red solid line). Parameters are  $\omega = 1.2$ ,  $M_0 = 1.1$ ,  $M_1 = 1.11$ ,  $\eta = 40\omega$ ,  $\omega_D = 1000\eta$ , and  $\hbar = k = 1$ .

expansion of the variances (2) and (3) in the strong coupling limit  $\omega_D \gg \eta/M \gg \omega$  yields [23,27]

$$\langle q^2 \rangle(\eta, M) = \frac{2\hbar}{\pi\eta} \ln \frac{\eta}{M\omega} + \frac{\pi\eta}{3\hbar M^2 \omega^4} (kT)^2 + \mathcal{O}(T^3), \quad (6)$$

$$\langle p^2 \rangle(\eta, M) = \frac{\hbar\eta}{\pi} \ln \frac{\omega_D M}{\eta} + \mathcal{O}(T^3). \quad (7)$$

With the help of the above expressions, we obtain

$$Q^{(M)} = (b_0 - b_1) \frac{\hbar\omega}{2\pi} \left[ 1 - \frac{\pi^2}{6a^2} \right], \quad (8)$$

$$\Delta S^{(M)} = \frac{1}{2} \left[ \ln \frac{\ln c_1}{\ln c_0} - \ln \frac{\ln b_0}{\ln b_1} - \frac{\pi^2}{24a^2} \left( \frac{b_0^2}{\ln b_0} - \frac{b_1^2}{\ln b_1} \right) \right], \quad (9)$$

where we have defined the dimensionless parameters,  $a = \beta\hbar\omega/2$ ,  $b_i = \eta/(M_i\omega)$ , and  $c_i = M_i\omega_D/\eta$ . In the low temperature, strong coupling regime,  $Q^{(M)}$  is always positive, that is, heat is absorbed by the system, while information can be erased,  $\Delta S^{(M)} < 0$ .

*Resolution of the paradox.* A key observation is that the Clausius inequality presupposes an initial Gibbs state [18]. We here resolve the paradox by treating the initial coupling as a thermodynamic transformation. By generalizing the method of *adiabatic switching*, well-known in chemistry [28], we evaluate entropy and heat resulting from the coupling of the isolated oscillator to the reservoir. The initial state of the system is now Gibbsian, in contrast to the previous case. When the coupling coefficient is quasistatically increased from 0 to  $\eta$ , the internal energy changes by  $\Delta U^{(C)} = U(\eta, M_0) - U(0, M_0)$ . The corresponding heat can be determined via the first law,  $Q^{(C)} = \Delta U^{(C)} - W^{(C)}$ , noting that the work required to couple the oscillator is given by the free energy difference  $W^{(C)} = \Delta F^{(C)} = F(\eta, M_0) - F(0, M_0)$  [22]. The latter can

be evaluated using the general form of the free energy of a quantum damped oscillator [23,27],

$$\beta F = \ln \Gamma\left(\frac{\beta \hbar \omega_D}{2\pi}\right) - \sum_{i=1}^3 \ln \Gamma\left(\frac{\beta \hbar \lambda_i}{2\pi}\right) - \ln\left(\frac{\beta \hbar \omega}{4\pi^2}\right), \quad (10)$$

where  $\Gamma$  denotes the Euler  $\Gamma$  function. In the absence of coupling ( $\eta = 0$ ), Eq. (10) reduces to  $F(0, M_0) = 1/\beta \ln[2 \sinh(\beta \hbar \omega/2)]$ . In the low temperature, strongly damped limit, we find that  $Q^{(C)} < 0$ , indicating that heat is dissipated into the environment:

$$Q^{(C)} = -\frac{\hbar \omega b_0}{2\pi} \left[1 - \frac{\pi^2}{6a^2}\right]. \quad (11)$$

The approximate, low temperature entropy change,  $\Delta S^{(C)} = S(\eta, M_0) - S(0, M_0)$ , can be computed in a similar way as before and reads

$$\Delta S^{(C)} = 1 + \frac{1}{2} \left[ \ln \frac{2}{\pi^2} + \ln \ln c_0 + \ln \ln b_0 + \frac{\pi^2 b_0^2}{24a^2 \ln b_0} \right]. \quad (12)$$

It is worth noticing that, contrary to the case of the mass variation [Eq. (9)], the entropy change induced by the coupling to the reservoir is here positive,  $\Delta S^{(C)} > 0$ .

We next consider the combined state transformation that consists of the initial coupling to the reservoir followed by the variation of the mass. We accordingly define the total entropy change  $\Delta S = \Delta S^{(M)} + \Delta S^{(C)}$  and the total heat  $Q = Q^{(M)} + Q^{(C)}$ . We again introduce a quantity  $\Delta = Q - kT \Delta S = \Delta^{(M)} + \Delta^{(C)}$  which, using Eqs. (8), (9) and (11), (12), we can write as

$$\Delta = -\frac{\hbar \omega}{2a} - \frac{\hbar \omega b_1}{2\pi} \times \left\{ 1 + \frac{\pi}{b_1 2a} \left[ \ln \frac{2}{\pi^2} + \ln \ln c_1 + \ln \ln b_1 \right] - \frac{\pi^2}{6a^2} \right\}. \quad (13)$$

In the limit of large  $a$ ,  $b$ , and  $c$ , this expression is always negative (see also Fig. 1). Thus, no violation of the Clausius inequality occurs in the low temperature, strong coupling regime, in agreement with standard thermodynamics. We can now understand the apparent violation found in Refs. [5–10] by comparing the change of entropy and heat during the two state transformations (see Fig. 2). The low temperature mass variation is characterized by a negative entropy change and a positive heat,  $\Delta S^{(M)} < 0$ ,  $Q^{(M)} > 0$ , which together lead to the breakdown of the Clausius inequality. On the other hand, the situation is exactly opposite for the initial reservoir coupling where  $\Delta S^{(C)} > 0$  and  $Q^{(C)} < 0$ . Figure 2 shows that in the limit of strong coupling, the latter contributions are much larger than those coming from the mass variation. They can therefore not be neglected as done so far in Refs. [5–10].

Deeper insight into the foregoing discussion can be gained by using the concept of Hamiltonian of mean force. We express

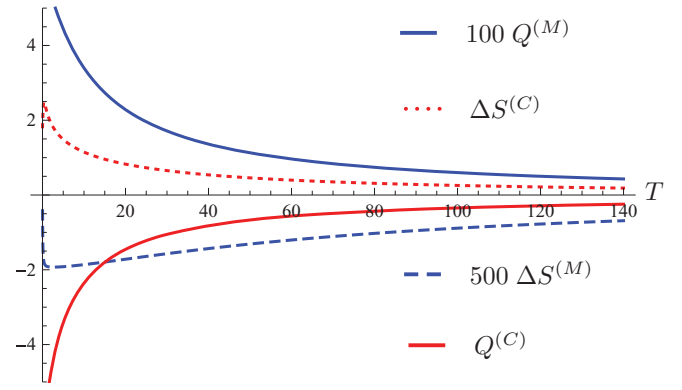


FIG. 2. (Color online) Entropy and heat during the initial quasistatic coupling to the reservoir,  $\Delta S^{(C)}$  (red dotted line) and  $Q^{(C)}$  (lower red solid line), and during a quasistatic mass variation,  $\Delta S^{(M)}$  (blue dashed line) and  $Q^{(M)}$  (upper blue solid line), as a function of temperature. Changes for the initial coupling have opposite signs and much larger amplitudes than those for the mass variation. Parameters are the same as in Fig. 1.

the reduced density operator of the system in the form  $\rho_S = \exp[-\beta(H_S^* - F)]$ , where [29]

$$H_S^* = -\frac{1}{\beta} \ln \frac{\text{Tr}_B \exp(-\beta H)}{\text{Tr}_B \exp(-\beta H_B)} \quad (14)$$

is the quantum Hamiltonian of mean force and  $F = -1/\beta \ln \text{Tr}_S \exp(-\beta H_S^*)$  the free energy of the system. The quantity  $\Delta H_S = H_S^* - H_S$  vanishes for vanishing reservoir coupling and thus quantifies the deviation from a Gibbs state. It is simply related to the initial thermodynamic change of the system. We first note that the von Neumann entropy of the system is given by  $S = \beta(U - F + \langle \Delta H_S \rangle)$ . The heat exchanged with the reservoir during the initial coupling is then  $Q^{(C)} = kT \Delta S^{(C)} - \langle \Delta H_S \rangle$  or, in other words,  $\Delta^{(C)} = -\langle \Delta H_S \rangle$ . We therefore find that  $\Delta = \Delta^{(M)} - \langle \Delta H_S \rangle$ . This result, valid for any quantum dissipative system, shows that the difference between the combined state transformation and the mass variation alone is just the difference between the Hamiltonian of mean force and the bare Hamiltonian of the system. It is worth stressing that for the quantum harmonic oscillator,  $\langle \Delta H_S \rangle$  is a function of the variances (2) and (3) of the reduced stationary state and can therefore be determined experimentally.

The following physical picture thus emerges from our analysis: In the limit of vanishing system-reservoir coupling, the stationary state of the system is Gibbsian and the thermodynamic cost of the coupling to the reservoir is negligible,  $\Delta = \Delta^{(M)}$ . In the opposite limit of strong coupling, the stationary state becomes non-Gibbsian, but the thermodynamic contributions of the initial coupling are important and hence need to be fully included,  $\Delta = \Delta^{(M)} - \langle \Delta H_S \rangle$ . In both cases, as we have just proved, the ordinary Clausius inequality  $\Delta \leq 0$  holds. We note that the Clausius inequality reduces to an equality when the system is in a Gibbs state at all times [18]: Even for quasistatic transformations, this is only achieved in the high temperature, weak coupling limit (see Fig. 1). An effective Clausius inequality has lately been derived by introducing an effective mass and spring constant for the oscillator, as well as an effective temperature, which differs

from that of the reservoir [30]; the latter is at variance with the zeroth law.

*Landauer's principle.* Let us finally derive the Landauer bound from the quantum Clausius inequality. We consider an isolated system with two stable states that are used to encode one bit of information (e.g., a symmetric double-well potential with high energy barrier). The system is initially in equilibrium at temperature  $T$  and the two states are occupied with equal probability. We reset the memory by first coupling it to the reservoir and then modulating the potential in order to bring the system with probability one into one of its states [2]. The von Neumann entropy of the system is hence  $\ln 2$  before the coupling to the reservoir and zero after complete erasure. From the Clausius inequality, we then find that the dissipated heat obeys  $Q_{\text{dis}} = -Q \geq -kT \Delta S = kT \ln 2$ . This is Landauer's erasure principle.

*Conclusion.* Our findings emphasize the crucial role of system-reservoir interactions in the thermodynamic

description of quantum systems; a low temperature investigation can therefore only be consistent if they are fully taken into account. We have derived detailed expressions for the change of entropy and heat induced by a quasistatic coupling to the reservoir. We have further shown that their relative contributions grow with increasing interaction strength and even dominate in the strong coupling limit, thus safeguarding the validity of Clausius' inequality. In contrast to other approaches [31], we have here used standard thermodynamic definitions of heat and work. Contrary to previous claims that quantum correlations undermine Landauer's erasure principle, we have demonstrated that the principle does hold when the generation of these correlations is properly included.

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